Capas

Para $n \in \mathbb{N}$, sean los conjuntos:

$$F_n = \{(x_0, x_1, \cdots, x_{n-1}) \in \{0,1\}^n | x_0 \geq x_1 \geq \cdots \geq x_{n-1}\}$$

$$G_n = \{(x_0, x_1, \cdots, x_{n-1}) \in \{0,1\}^n | x_0 + x_1 + \cdots + x_{n-1} = 1\}$$

Sea la función $H_1: F_1 \to G_2$ que se define así:

$$0 \mapsto (0,1) = \left(f_{1,0}(0), f_{1,1}(0)\right)$$

$$1 \mapsto (1,0) = \left(f_{1,0}(1), f_{1,1}(1)\right)$$

$$H_1(x_0) = \left(f_{1,0}(x_0), f_{1,1}(x_0)\right)$$

donde,

$$f_{1,0}(x_0) = x_0$$

$$f_{1,1}(x_0) = 1 - x_0 = 1 - f_{1,0}(x_0)$$

Sea la función $H_2: F_2 \to G_3$ que se define así:

$$(0,0) \mapsto (0,0,1) = \left(f_{2,0}(0,0), f_{2,1}(0,0), f_{2,2}(0,0)\right)$$

$$(1,0) \mapsto (0,1,0) = \left(f_{2,0}(1,0), f_{2,1}(1,0), f_{2,2}(1,0)\right)$$

$$(1,1) \mapsto (1,0,0) = \left(f_{2,0}(1,1), f_{2,1}(1,1), f_{2,2}(1,1)\right)$$

$$H_2(x_0, x_1) = \left(f_{2,0}(x_0, x_1), f_{2,1}(x_0, x_1), f_{2,2}(x_0, x_1)\right)$$

donde,

$$f_{2,0}(x_0, x_1) = x_1 = f_{2,0}(x_1)$$

$$f_{2,1}(x_0, x_1) = (1 - x_1)x_0 = \left(1 - f_{2,0}(x_0, x_1)\right) f_{2,0}(x_0)$$

$$f_{2,2}(x_0, x_1) = (1 - x_1)(1 - x_0) = \left(1 - f_{2,0}(x_0, x_1)\right) f_{2,1}(x_0)$$

Sea la función $H_3: F_3 \to G_4$ que se define así:

$$(0,0,0) \mapsto (0,0,0,1) = \left(f_{3,0}(0,0,0), f_{3,1}(0,0,0), f_{3,2}(0,0,0), f_{3,3}(0,0,0)\right)$$

$$(1,0,0) \mapsto (0,0,1,0) = \left(f_{3,0}(1,0,0), f_{3,1}(1,0,0), f_{3,2}(1,0,0), f_{3,3}(1,0,0)\right)$$

$$(1,1,0) \mapsto (0,1,0,0) = \left(f_{3,0}(1,1,0), f_{3,1}(1,1,0), f_{3,2}(1,1,0), f_{3,3}(1,1,0)\right)$$

$$(1,1,1) \mapsto (1,0,0,0) = \left(f_{3,0}(1,1,1), f_{3,1}(1,1,1), f_{3,2}(1,1,1), f_{3,3}(1,1,1)\right)$$

$$H_3(x_0,x_1,x_2) = \Big(f_{3,0}(x_0,x_1,x_2), f_{3,1}(x_0,x_1,x_2), f_{3,2}(x_0,x_1,x_2), f_{3,3}(x_0,x_1,x_2)\Big)$$

donde,

$$f_{3,0}(x_0,x_1,x_2) = x_2 = f_{3,0}(x_1,x_2)$$

$$f_{3,1}(x_0,x_1,x_2) = (1-x_2)x_1 = \Big(1 - f_{3,0}(x_0,x_1,x_2)\Big) f_{3,0}(x_0,x_1)$$

$$f_{3,2}(x_0,x_1,x_2) = (1-x_2)(1-x_1)x_0 = \Big(1 - f_{3,0}(x_0,x_1,x_2)\Big) f_{3,1}(x_0,x_1)$$

$$f_{3,3}(x_0,x_1,x_2) = (1-x_2)(1-x_1)(1-x_0) = \Big(1 - f_{3,0}(x_0,x_1,x_2)\Big) f_{3,2}(x_0,x_1)$$

Para $k \in \mathbb{N} \cup \{0\}$ (con $k \leq n$) sea la función $f_{n,k}: \{0,1\}^n \to \{0,1\}$ que se define así:

$$f_{n,k}(x_0,\cdots,x_{n-1}) = \begin{cases} x_{n-1}, & n = 1, k = 0 \\ 1 - f_{n,k-1}(x_0,\cdots,x_{n-1}), & n = 1, k = 1 \\ f_{n,k}(x_0,\cdots,x_{n-1}), & n > 1, k = 0 \\ \big(1 - f_0(x_0,\cdots,x_{n-1})\big) f_{n,k-1}(x_0,\cdots,x_{n-2}), & n > 1, k > 0 \end{cases}$$

Esto es,

$$f_{n,k}(x_0,\cdots,x_{n-1}) = \begin{cases} x_{n-1}, & n \geq 1, k = 0 \\ 1 - x_{n-1}, & n = 1, k = 1 \\ (1 - x_{n-1}) f_{n,k-1}(x_0,\cdots,x_{n-2}), & n > 1, k > 0 \end{cases}$$

Finalmente, sea la función $H_n: F_n \to G_{n+1}$ que se define así:

$$H_n(x_0,\cdots,x_{n-1}) = \Big(f_{n,0}(x_0,\cdots,x_{n-1}),\cdots, f_{n,n}(x_0,\cdots,x_{n-1})\Big)$$

Proyectores

Sean $n \in \mathbb{N}$ y $k \in \mathbb{N} \cup \{0\}$ con $k \leq n$, se define la función $P_{n+1,k}: \mathbb{R}^{n+1} \to \mathbb{R}$, así:

$$P_{n+1,k}(y_0,\cdots,y_n) = (y_0,\cdots,y_n) \cdot H_n(x_0,\cdots,x_{n-1}) = y_k$$

donde,

$$x_0 + x_1 + \cdots + x_{n-1} = n - k$$